**Chapter 13. Symmetry Groups**

# Groups

# Definitions:

A **group** is a set G with an operation ** that is closed and associative, has an identity e, and every element *g* has an inverse *g*-1 such that *gg*-1 = e = *g*-1*g.*

A group G is **Abelian** if it is commutative: *gh* = *hg* for all *g*, *h* in G.

A **subgroup** is a subset of G that is a group under **.

Let H be a subgroup of G. A **coset of H** is a set H*g* = {*h g*: hH}, where *g*G. The only coset of H that is a group is the set H itself: H = He where e is the identity element. The cosets of H form a partition of G.

A **normal subgroup** is a subgroup H that satisfies *g*H = H*g* for all *g* in G, or equivalently H = *g*-1**H*g*.

A group is **simple** if it contains no non-trivial normal subgroup. The simple groups are the fundamental “building blocks” of more complex groups.

Theorem. There are precisely 4 **classical** and 5 **exceptional** simple Lie groups.

* Classical Families: Am, Bm, Cm, Dm having dimensions , , , and , respectively where .
* Exceptional Groups: E6, E7, E8, F4, G2 of dimension 78, 133, 248, 52, and 14 respectively

Theorem. The simple finite groups have been classified into classical and exceptional groups. The largest exceptional group has ≈ 1060 elements and is known as **the** **monster.**

Definition. The **Product Group** of groups G and H is **GxH** =  with group operation .

Definition. Let N be a subgroup of G. The **Factor Space** **G/N** is the collection of cosets N*g* along with the operation (N*g*1)**(N*g*2) = N**(*g*1 *g*2).

Theorem. If N is normal then G/N is a group, called the **Factor Group.**

Theorem. [13.10]  .

# Symmetries of a Square

1

*a*

*b*

*c*

*d*

i

*d*

*a*

*b*

*c*

-1

*c*

*d*

*a*

*b*

-i

*b*

*c*

*d*

*a*

C

*a*

*d*

*c*

*b*

iC

*b*

*a=a*

*d*

*c*

-C

*c*

*b*

*a*

*d*

-iC

Ci

*d*

*c*

*b*

*a*

Definitions**:**

**Non-reflecting Group:** < i > = {1, i, -1, -i}

**Reflecting Group:** < i, C > = {1, i, -1, -i, C, iC, -C, -iC = Ci}

**C** is complex conjugation: . **1** is the null rotation, which is the group identity element. ***i*** is the 90° counter-clockwise rotation of the square

Convention: a b means b acts first.

A subgroup of a symmetry group is called a **reduced symmetry group**.

Examples:

Normal subgroups of < i, C > :

{1, -1, C, -C} , {1, -1}, {1, -1}

Non-normal subgroups of < i, C > :

{1, -C}, {1, iC}, {1, C}

For example, {1, C} i = {i, Ci} ≠ {i, -Ci} = i {1, C}

Example [13.6]: Reduced symmetry groups can be generated using one or more arrows.

1

*a*

*b*

*c*

*d*

i

*d*

*a*

*b*

*c*

-1

*c*

*d*

*a*

*b*

-i

*b*

*c*

*d*

*a*

C

*a*

*d*

*c*

*b*

iC

*b*

*a=a*

*d*

*c*

-C

*c*

*b*

*a*

*d*

-iC

Ci

*d*

*c*

*b*

*a*

{1, C} is a reduced symmetry group

1

*a*

*b*

*c*

*d*

i

*d*

*a*

*b*

*c*

-1

*c*

*d*

*a*

*b*

-i

*b*

*c*

*d*

*a*

C

*a*

*d*

*c*

*b*

iC

*b*

*a=a*

*d*

*c*

-C

*c*

*b*

*a*

*d*

-iC

Ci

*d*

*c*

*b*

*a*

{1, iC} is a reduced symmetry group

1

*a*

*b*

*c*

*d*

i

*d*

*a*

*b*

*c*

-1

*c*

*d*

*a*

*b*

-i

*b*

*c*

*d*

*a*

C

*a*

*d*

*c*

*b*

iC

*b*

*a=a*

*d*

*c*

-C

*c*

*b*

*a*

*d*

-iC

Ci

*d*

*c*

*b*

*a*

{1, -1, C, -C} is a reduced symmetry group

# Symmetries of a Sphere

Definitions:

A group G whose underlying set is continuous is called a **Lie Group.**

**SO(3)** is the group of non-reflective symmetries of a 3-sphere

**O(3)** is the **Orthogonal Group**. It consists of both the reflective and non-reflective symmetries of a sphere.

O(3) = SO(3)  T, the disjoint union of O(3) with the coset of reflective symmetries

 where **R** is the reflection operator on the sphere.

Recall problem [12.7]: SO(3) is group isomorphic to the solid sphere  of radius ** with antipodal points identified.

Theorem. (Problem [13.7]) SO(3) and {1, R} are the only normal subgroups of O(3), where **1** is the null rotation. (Penrose overlooked that the latter group is normal.)

Examples. Reduced Symmetry Groups

The set of rotations that fix a point on the sphere forms a non-normal subgroup. It is the set of rotations having the arrow as its axis.

Marking the sphere with vertices of a regular polyhedra reduces to the finite group of rotations of the sphere that take each vertex to one of the others. Such reduced symmetry groups are non-normal.

# Linear Transformations and Matrices

Definition. Let V and W be vector spaces.

* *f* : V W is a **homomorphism** if it preserves the vector space structure:
  + *f*  (*a u* + *b v*) = *a f* (*u*) + *b f* (v) for all vectors *u* and *v* and scalars *a* and *b*.
* **Hom(V,W)** is the set of homomorphisms from V to W.
* **A(V)**= Hom(V,V).
* A **linear transformation** is a member T A(V).
  + That is, a linear transformation is a function  such that T(*au*+*bv*) = *a*T*u* + *b*T*v*.

Theorem. [13.12, 13.13] Let V = ℝ3, using (*x1*, *x2*, *x2*) instead of (*x*, *y*, *z*). Then a linear transformation T takes the form .

Note. Linear transformations are represented by matrices:



**x**

**x**

In diagrammatic form this is 

Theorem. If R = S T then . That is, the composition, R, of 2 linear transformations is the result of matrix multiplication of S and T. In diagrammatic notation:

R = = = S T

Example. T I = T = I T is written in diagrammatic form as = = .

and, in ℝ3, I =  where *a*, *b* range over {1, 2, 3}.

Definitions. A linear transformation T is **singular** if Dim(TV) < Dim W; that is, T is not *onto*.

Theorem. [13.17] T is singular iff  *v* ≠ 0 such that T*v* = 0.

Corollary. [Bud] T is 1-1 iff T is non-singular iff T is onto.

Proof: T is 1-1  T(*v* – *w*) = T(*v*) – T(*w*) ≠ 0 

 T is non-singular  T is onto.

(\*) Set *v* = 3*u* and *w* = 2*u*.  

Theorem. [13.18] If T is nonsingular, then it has an inverse T-1.

-1

*n*

Theorem. [13.19] T-1 = =

Definition. The **transpose** of the matrix T = (T*ij*) is the matrix **TT** = (Tji).

Definition. A matrix T is **orthogonal** if T-1 = TT.

**Determinants and Traces**

Definition. Det T = .

Theorem. [Bud] Det T =  (the normal definition of Det)

Proof. Let P1*…n* be the set of permutations of (1, …, *n*).

Det T = 

= 

(Replace Einstein notation.)

= 

(Replace ** by **  ** \* in  and T. The double sum over ** and ** \* is unchanged, in both expressions stepping over all permutations of (1, …, *n*), and the exponents of  continue to match the subscripts of T.)

= 

(Re-order superscripts of  by applying an inverse ** permutation.)

= 

(This is just a simpler way to label the subscripts and superscripts of T. For example, if **\*(3) = 1 then .)

= 

= 



(See my solution to [13.21] for examples of this for *n* = 2 and 3.)

Theorem. [Bud] *n*!

Proof: Let P*a*…*g* be the set of permutations of (*a*, …, *g*). Then

= 

= **

= *n*!

(\*) ** is the composition of transmutations (i.e., of pairwise permutations).

Let  be a transmutation. Then





.

So, for any permutation **, we have

  

Theorem. [13.22]

Det AB =   

= DetA DetB

Theorem. (p.260 – no proof given) Matrix A is singular iff Det A = 0.

Proof: From [13.19], A is non-singular iff Det A ≠ 0. 

Definition. Vectors *v* and *w* are **orthogonal** if . That is, the angle between them is 90°.

Theorem. A matrix is orthogonal (i.e., TT = T-1) iff its column vectors are mutually orthogonal.

Example. Orthogonal 2 x 2 Matrices: A and B

*w* = (- Sin **, Cos **)

*v* = (Cos **, Sin **)

*w1* = (Sin **, - Cos **)

**

Let 

.

.

AT = A-1 :

A AT =  ✔

Similarly AT A = I ✔

So A is an orthogonal matrix ✔

Det A = Det AT = Cos2 ** + Sin2 ** = 1 ✔

The column vectors of A are orthogonal: *v*  *w* ✔

Let . Then B BT = I, Det B = Det BT = -1, and its column vectors *v* and *w*1 are orthogonal.

**

*w*

*v*

*u*

Examples. Orthogonal 3 x 3 Matrices: A, B, and C

Let  , , and  .

Let .

. A is orthogonal, its columns are orthogonal vectors, and its determinant is +1. ✔

Let . B is orthogonal and its determinant is -1. ✔

Let C be a ** -rotation of A about an axis :



It can be directly verified that C is an orthogonal matrix with mutually orthogonal column vectors and determinant +1. ✔

Definition. A **symmetry** of a vector space (V,+) is a transformation T : VV that is 1-1 and onto that preserves the vector space structure:

T(*a v* + *b w*) = *a* T*v* + *b* T*w*

Definition. The **General Linear Group GL(*n*)** is the group of symmetries of an *n*‑dimensional vector space.

Theorem. GL(*n*) is the group of non-singular (*n* x *n*) matrices.

Proof. Let T  GL(*n*). Since T(*a v* + *b w*) = *a* T*v* + *b* T*w*, T is a linear transformation. Were T singular, then by [13.17] Dim T V < *n*  T is not onto, a contradiction. Therefore T is a non-singular linear transformation. Thus in any basis, T is represented by a non-singular matrix. 

Definition. The **Special Linear Group SL(*n*)** is the subset of GL(*n*) having determinant = 1.

Theorem. SL(*n*) is a normal subgroup of GL(*n*).

Proof. First, SL(*n*) is a group:

Closed: If S1, S2  SL(*n*), then Det(S1 S2) = Det(S1) Det(S2) = 1

 S1 S2  SL(*n*).

Identity: Det(I) = 1  I  SL(*n*)

Inverse: 1 = Det(I)= Det(S1 S1-1) = Det(S1) Det(S1-1) = Det(S1-1)

 S1-1  SL(*n*)

Also, SL(*n*) is normal:

Let S  SL(*n*) and G  GL(*n*). Then

Det(G-1 S G) = Det(G-1) Det(S) Det(G) = Det(G-1) Det(G)

= Det(G G-1) = Det(I) = 1

 G-1 S G  SL(*n*)  

The groundwork has now been laid to introduce the table, below, that shows the relationships between GL(3), O(3), SL(3), general linear transformations, orthogonality, determinants, and symmetries. The table shows that SL(3)  O(3)  GL(3) **A**(ℝ3), and GL(3) is both the set of symmetries of ℝ3 and the set of non-singular matrices. It also shows that the orthogonal group O(3) is a proper subset of the set of orthogonal matrices (shaded blue).

In general, non-singular matrices squeeze and stretch the unit sphere (or the reflected sphere) into an ellipsoid. However, singular matrices are more severe. They squash the unit sphere down to a 2-dimensional circle or ellipse or even to a line or a point.

Only orthogonal matrices preserve the sphere without squeezing or stretching any portion of it. This is achieved by limiting its operation to rotations and reflections. However, if determinant ≠ ±1 then orthogonal matrices also expand or contract the sphere.

Non-orthogonal matrices also squeeze, stretch, or preserve the sphere but not as rotations. Rather, the matrix columns would contain non-orthogonal vectors. In such a case the angle between the 1st and 2nd column vectors might be less than 90°, squeezing the sphere along associated plane. The angle between the 2nd and 3rd vectors would then be greater than 90°, stretching the sphere along that plane.

**A(**ℝ3**) = 3 x 3 Matrices**

|  |  |  |  |
| --- | --- | --- | --- |
| **Determinant** | **Orthogonal** | **Sphere maps to a …** | **Matrix Type** |
| 0 | Yes | Circle or line or point | Singular |
| No | Ellipse or line or point |
| Between  -1 and 0 | Yes | Contracted reflected sphere | **GL(3**)  Non-singular  Symmetries of ℝ3 |
| No | Contracted reflected ellipsoid |
| Between  0 and +1 | Yes | Contracted sphere |
| No | Contracted ellipsoid |
| -1 | Yes | Reflected sphere  O(3) |
| No | Reflected ellipsoid |
| +1 | Yes | SL(3) = sphere |
| No | Ellipsoid |
| < -1 | Yes | Expanded reflected sphere |
| No | Expanded reflected ellipsoid |
| > 1 | Yes | Expanded sphere |
| No | Expanded ellipsoid |

Matrices with positive determinant act on the sphere. Matrices with negative determinant behave exactly the same but act on the reflected sphere.

Definition. The **Trace** of A is Tr(A) = Tr .

Theorem: [Bud]

*c*

*b*

*a*

*t*

*s*

*r*

Tr  



Proof: Let P*ab…c* and P*rs…t* be the sets of permutations of (*a*,*b*,…,*c*) and (*r*,*s*,…,*t*),

*c*

*b*

*a*

*t*

*s*

*r*

respectively. Let B =

= .

Fix **. The only non-zero term in the sum is

.

I showed in Problem [13.22] that  for any fixed (*x,y*,…,*z*).

Thus, B = . This sum has *n*! terms composed of (*n* – 1)! terms equal to T*aa*, (*n* – 1)! terms equal to T*bb*, …, and (*n* – 1)! terms equal to T*cc*. So,

B = (*n* – 1)! (T*aa* + T*bb* + … + T*cc*) = (*n* – 1)! Tr (A) = (*n* – 1)! Tr

Similarly for the other figures. 

Theorem. [13.24]  if we ignore 2nd order and higher  terms.

Theorem. [13.25] Det *e*A = .

Definition. An **Eigenvector** is a non-zero vector *v* for which  such that . ** is called an **Eigenvalue.**

Note:  and so  is singular

Theorem. [13.26]  is a polynomial equation of degree *n*.

Definition. ****has** **multiplicity *r*** means that ** appears *r* times in the equation above. Eigenvalue multiplicities are called **degeneracies**in Quantum Mechanics.

Definition. The set of Eigenvectors corresponding to ** is a linear space called an **Eigenspace**.

Theorem. If *d* is the dimension of the Eigenspace of ** and *r* is the multiplicity of **then 1 ≤ *d≤***

Theorem. [13.27] Let  be the set of Eigenvalues of an *n* x *n* matrix T, and let *r i* be the multiplicity of * *******Then .

Corollary. A linear transformation T has at least 1 Eigenvector.

Theorem. [13.30] Suppose {*ek*} and {*fk*} are bases for a vector space V, and *fk* = T *ek*. Then

.

That is, the components of *fj* in basis {*ek*} are .

Theorem. [13.31] If the Eigenspace dimension of every multiple Eigenvector equals its multiplicity, then there is a basis for V composed of Eigenvectors, and the matrix of T in this basis is

.

The next theorem states that even when the hypothesis of the above theorem is not satisfied, the matrix of T can at least be written in upper triangular form.

Theorem. (Note 13.12): **Jordan Canonical Form:** Let  be the set of Eigenvalues of an *n* x *n* matrix T, and let *r i* be the multiplicity of **. Then there is a basis for V such that the matrix of T in this basis is

.